Data assimilation practicals

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10 August 2016

Data Assimilation
Research Centre

Particle filters

Bayes theorem

$$p(x \mid y) = \frac{p(x)p(y \mid x)}{p(y)} = \frac{p(x)p(y \mid x)}{\int p(x)p(y \mid x)\mathrm{d}x}$$

Particle filter basics

Write the prior pdf as a sum of delta functions

$$p(x) = \sum_{i=1}^{M} w_i \delta(x - x_i) \tag{1}$$

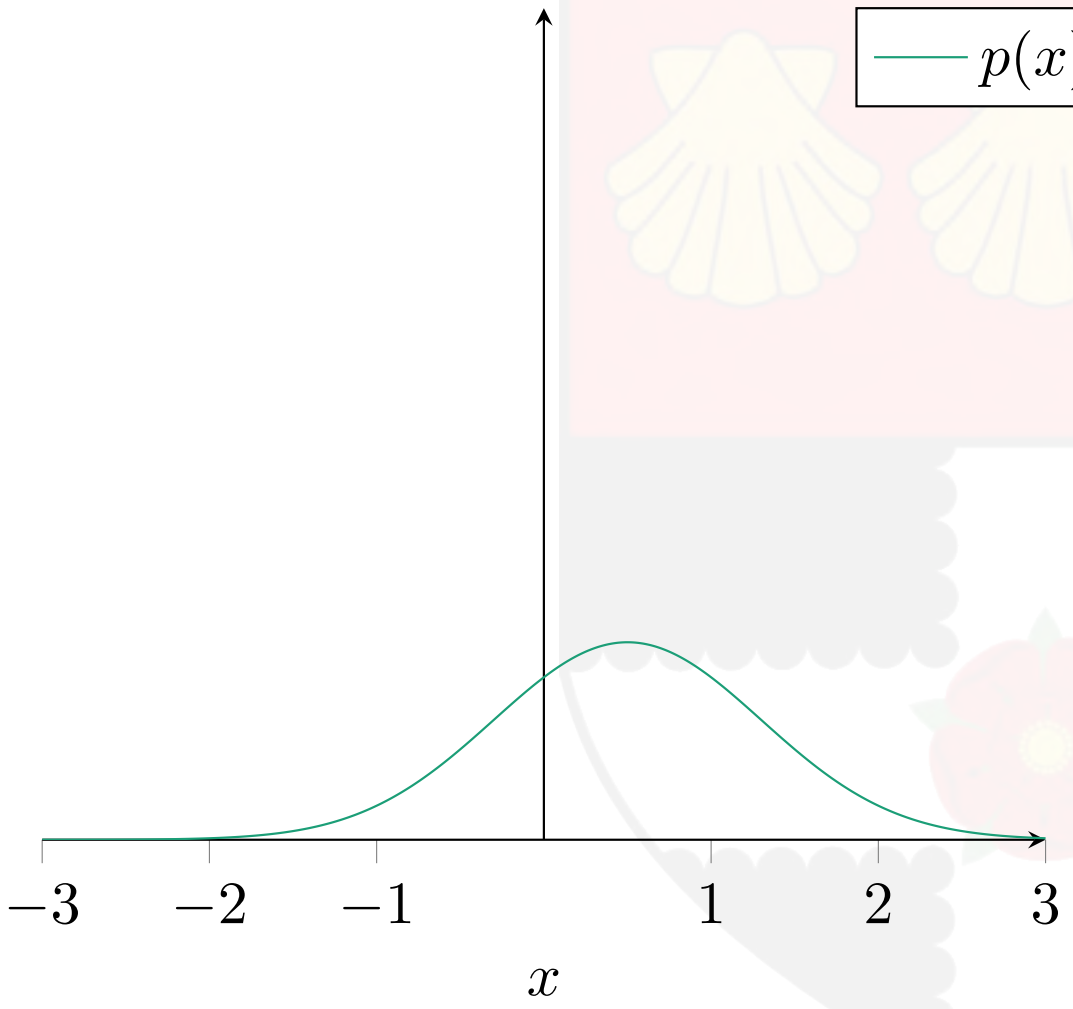
Take a pointwise evaluation of the likelihood $p(y \mid x)$.
Arrive at a new delta function representation of the posterior

$$p(x \mid y) = \sum_{i=1}^{M} \omega_i \delta(x - x_i) \tag{2}$$

such that

$$\omega_i = p(y \mid x_i) w_i \tag{3}$$

The point of particle filters



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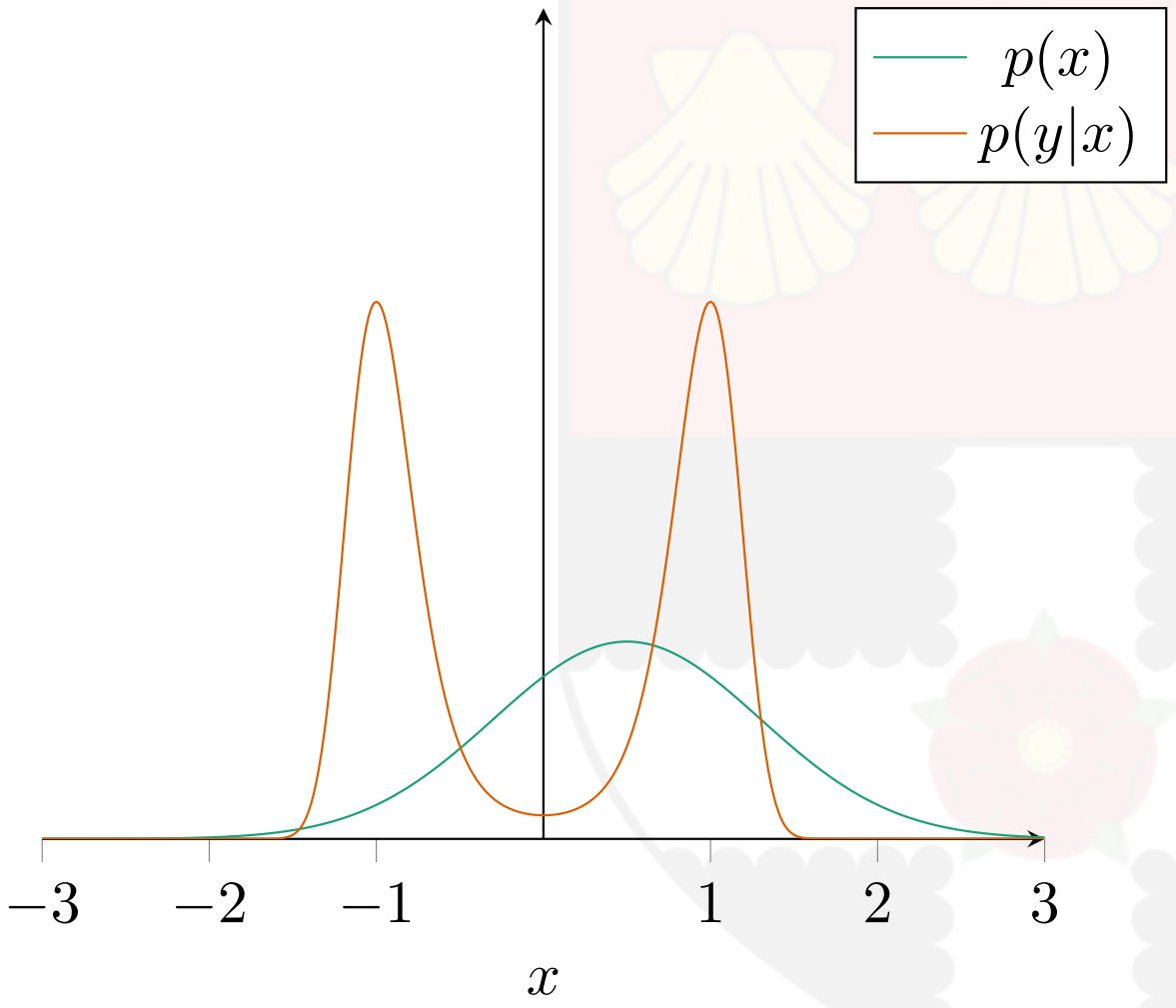

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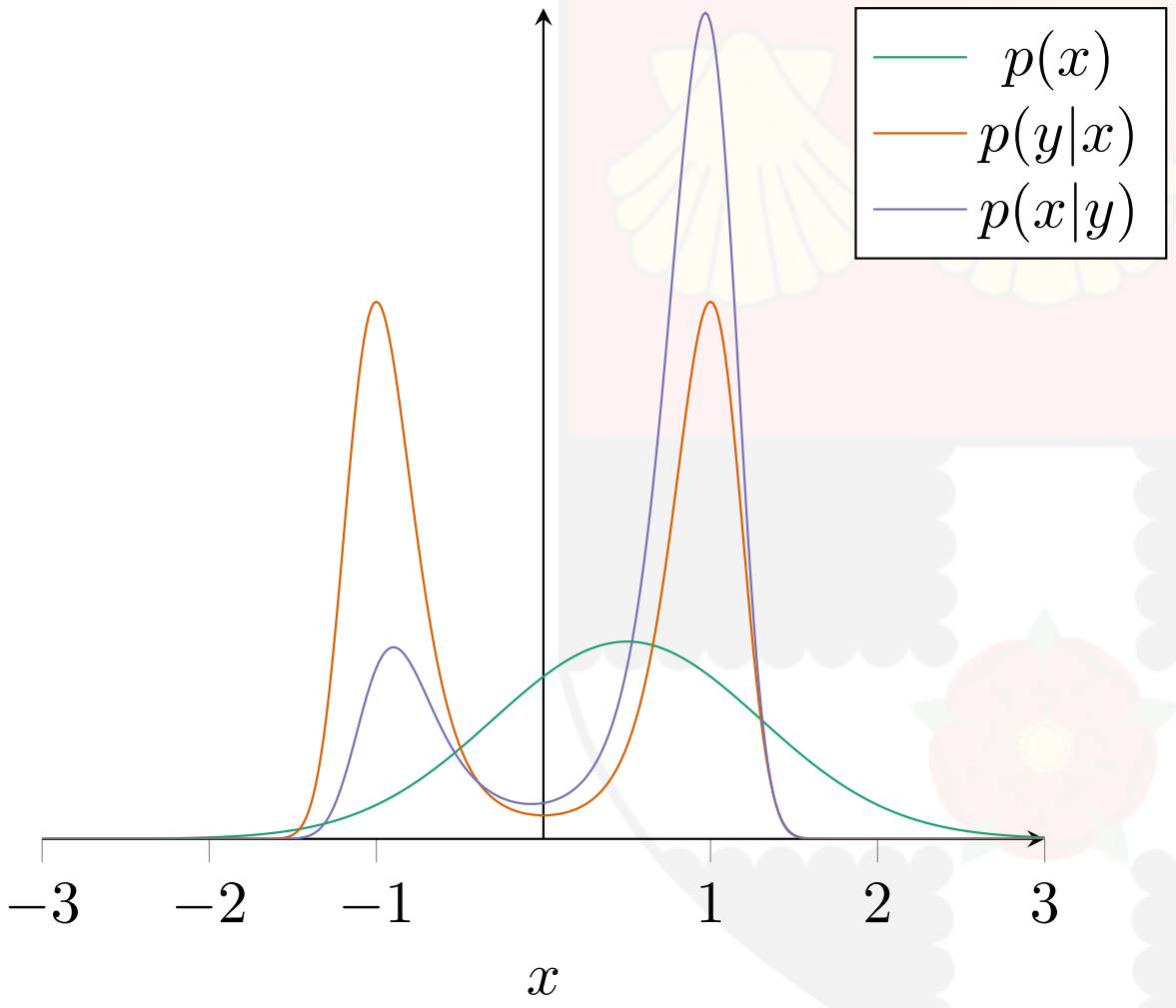

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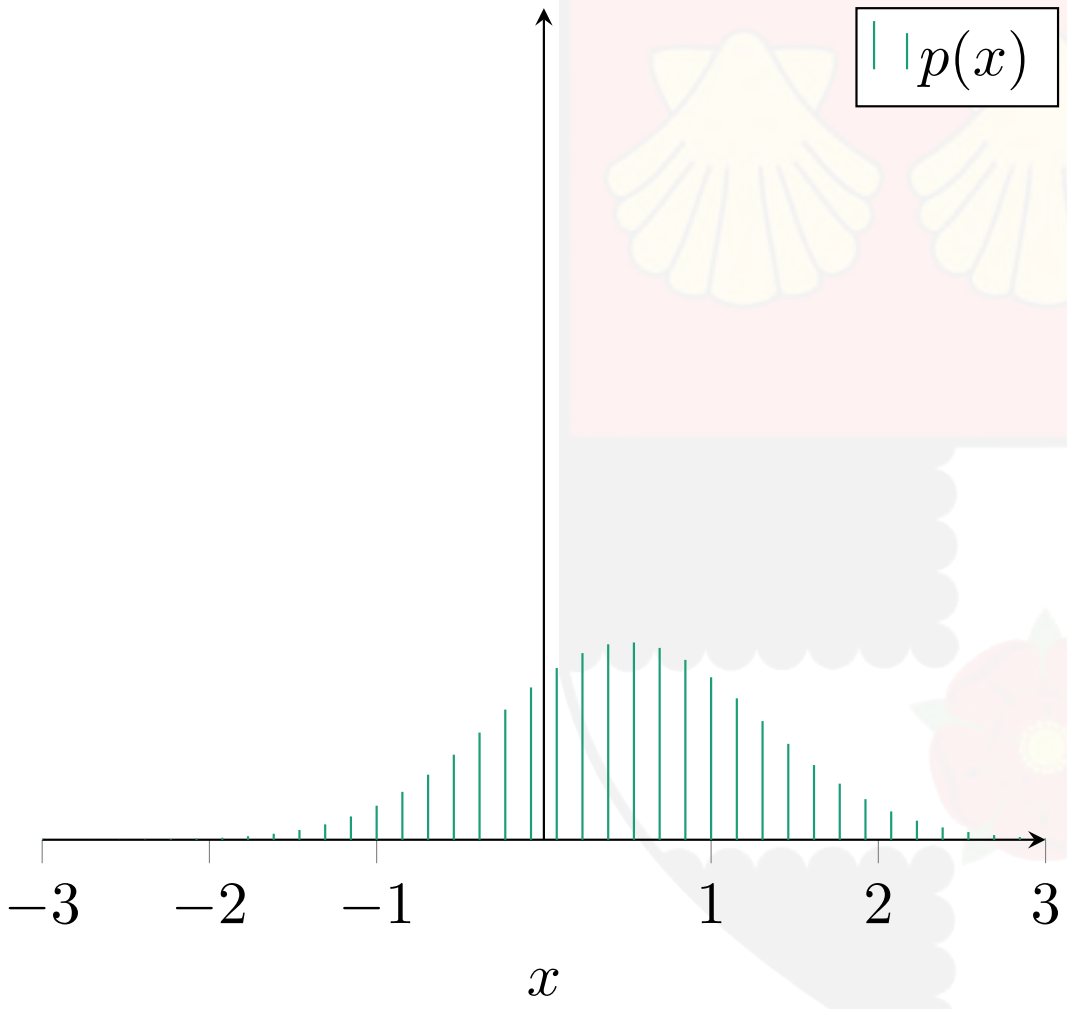

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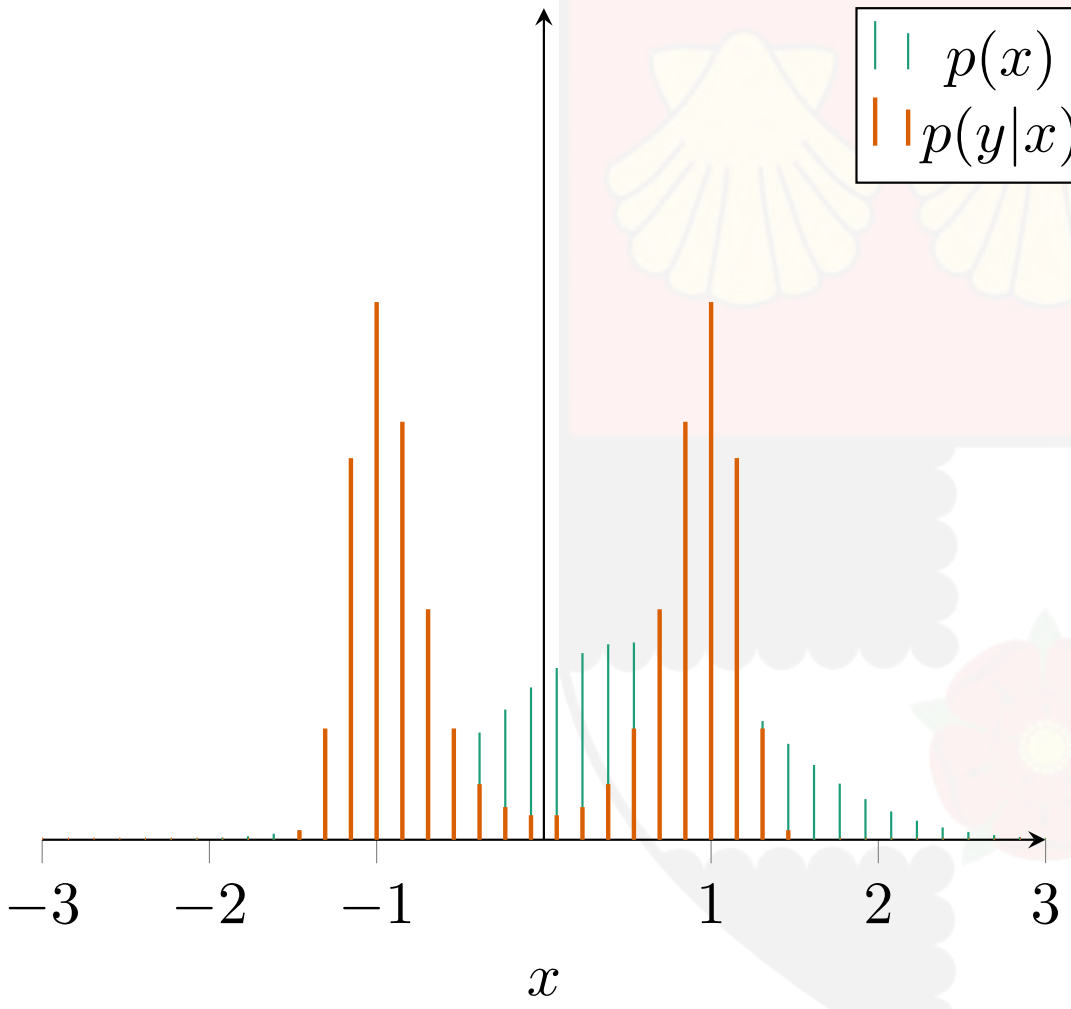

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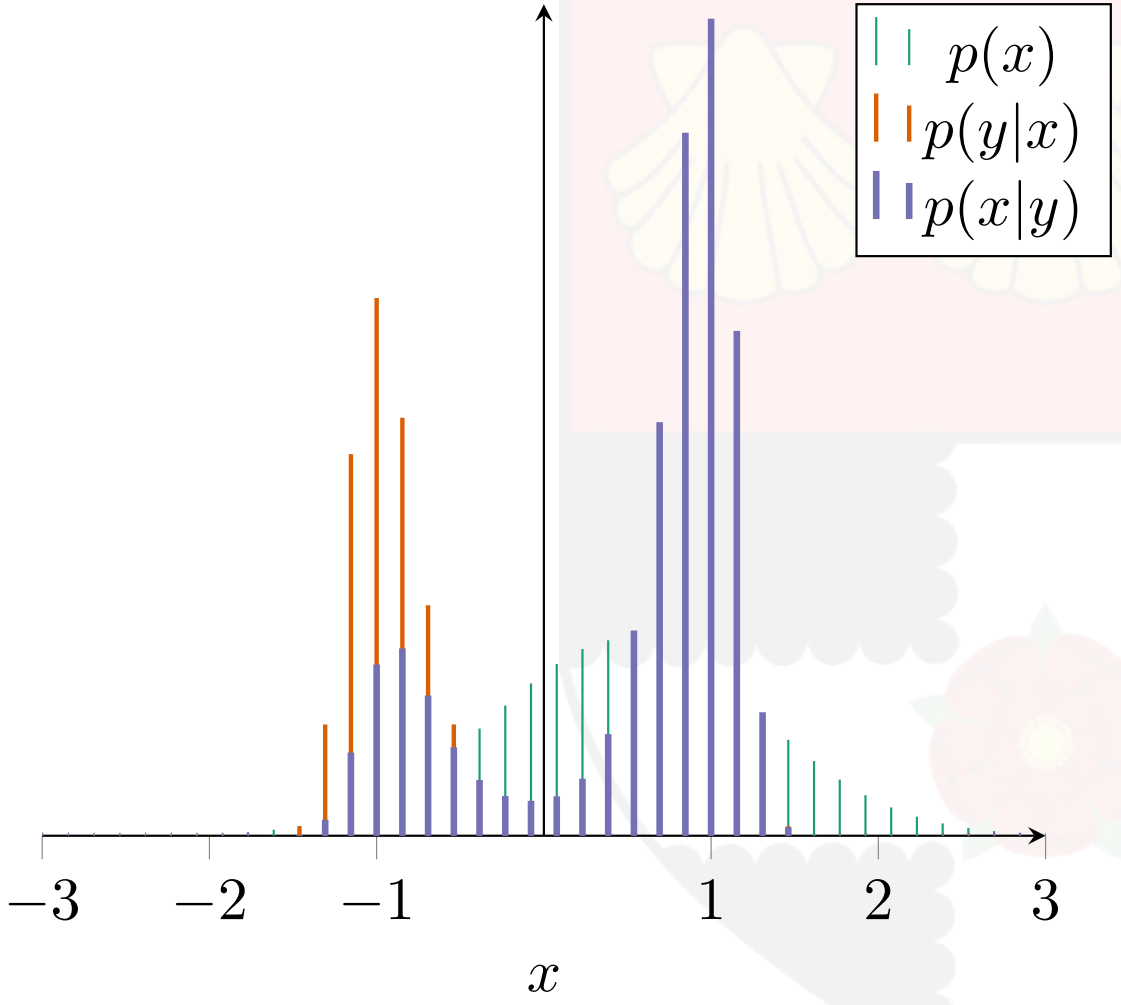

Filter degeneracy

2 simple definitions:

$$\omega_j = 1 \qquad \text{for some } j \in \{1, \ldots, M\} \tag{4}$$

or

$$x_i = x_j \qquad \text{for all } i, j \in \{1, \ldots, M\} \tag{5}$$

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Thus

$$p(x) = \sum_{i=1}^{M} w_i \delta(x - x_i) = \delta(x - x_j) \tag{6}$$

Which means

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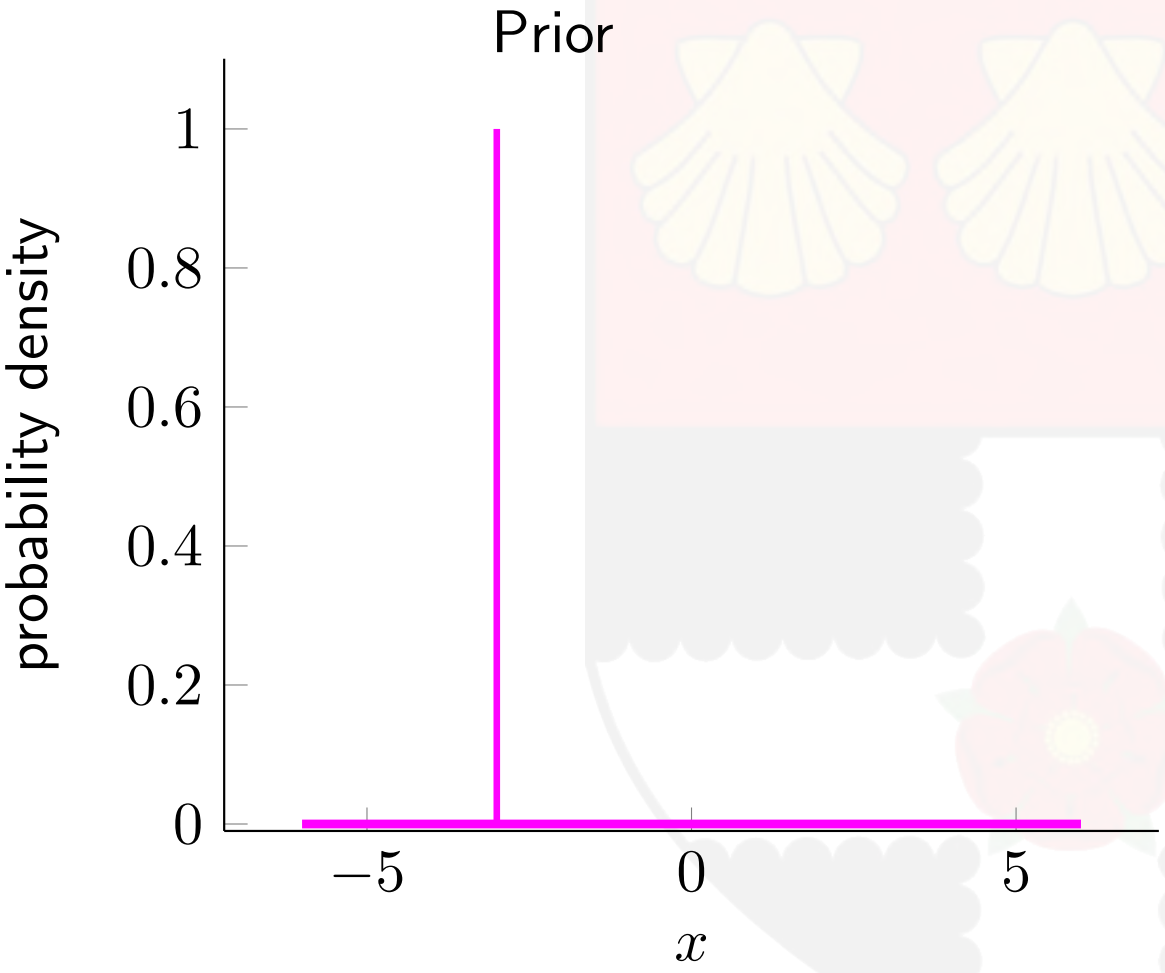
Which means

$$p(x \mid y) = \frac{p(x)p(y \mid x)}{\int p(x)p(y \mid x)\mathrm{d}x} = \frac{\delta(x - x_j)p(y \mid x)}{\int \delta(x - x_j)p(y \mid x)\mathrm{d}x} = \delta(x - x_j) \tag{7}$$

Filter degeneracy in pictures



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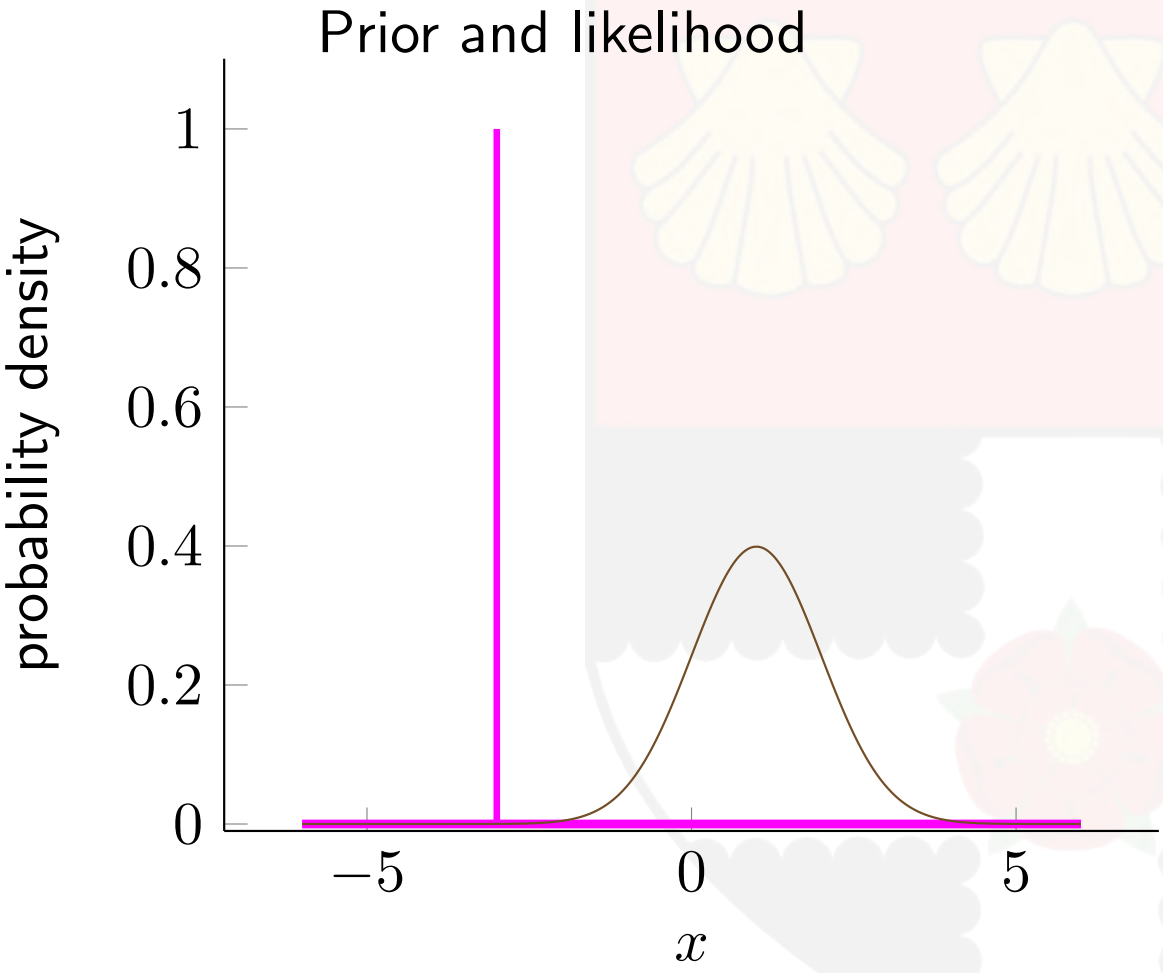

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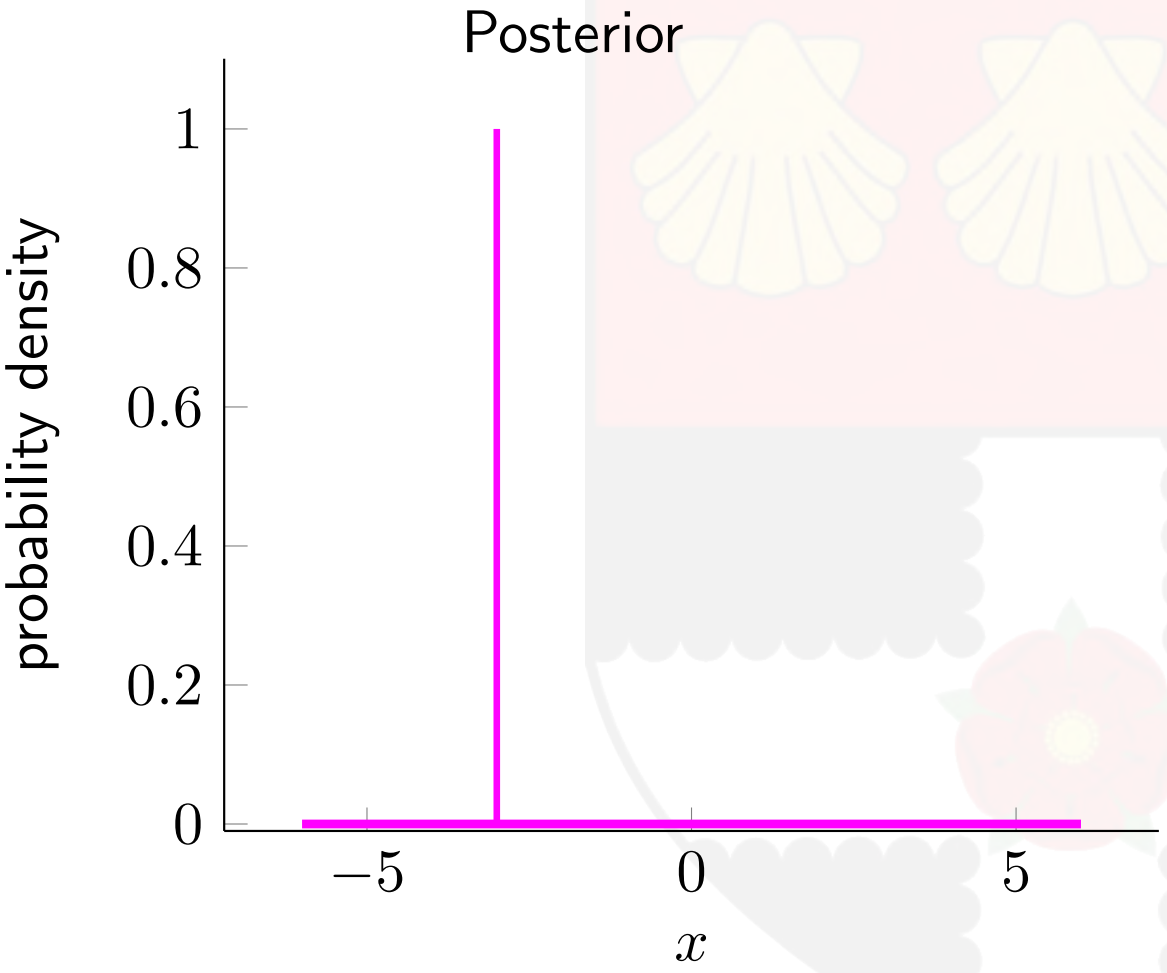

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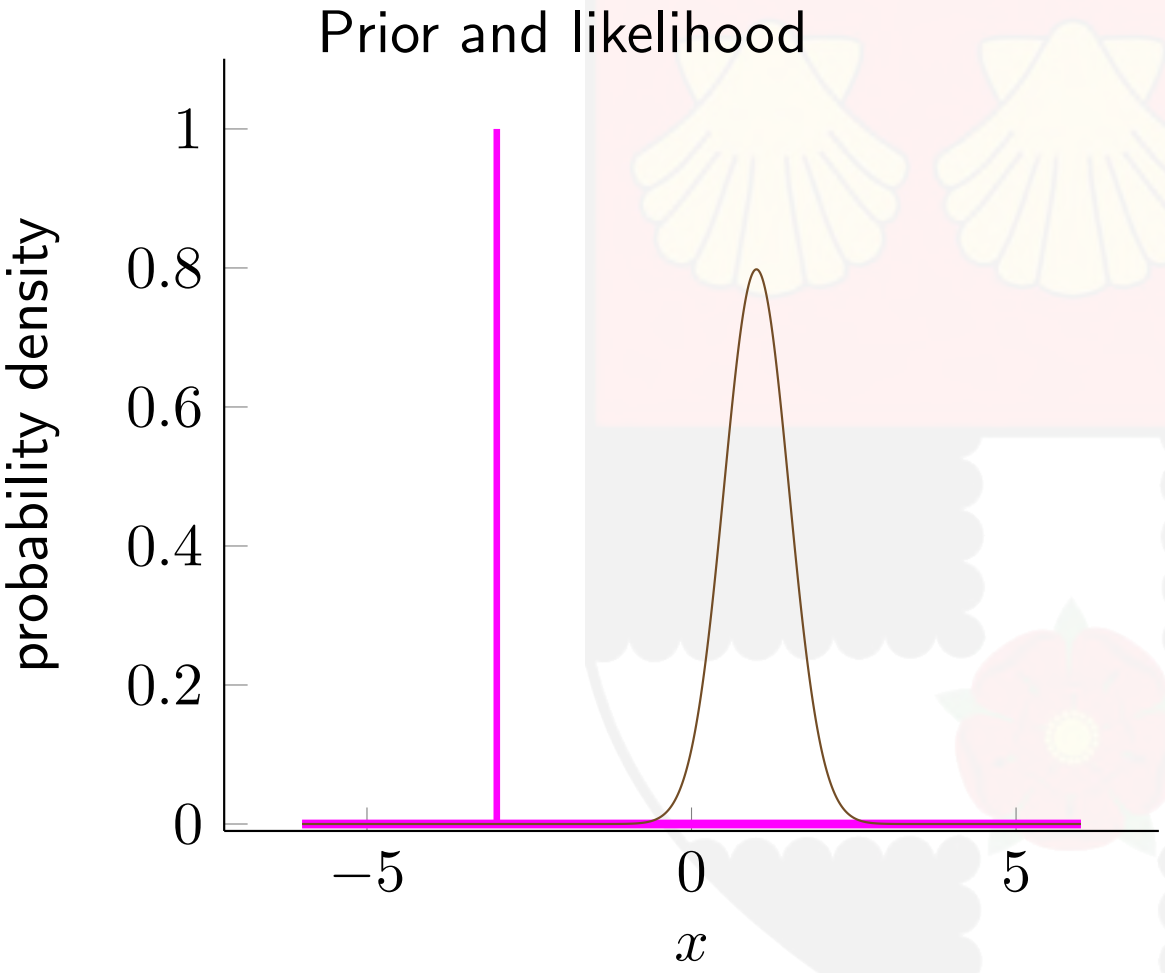

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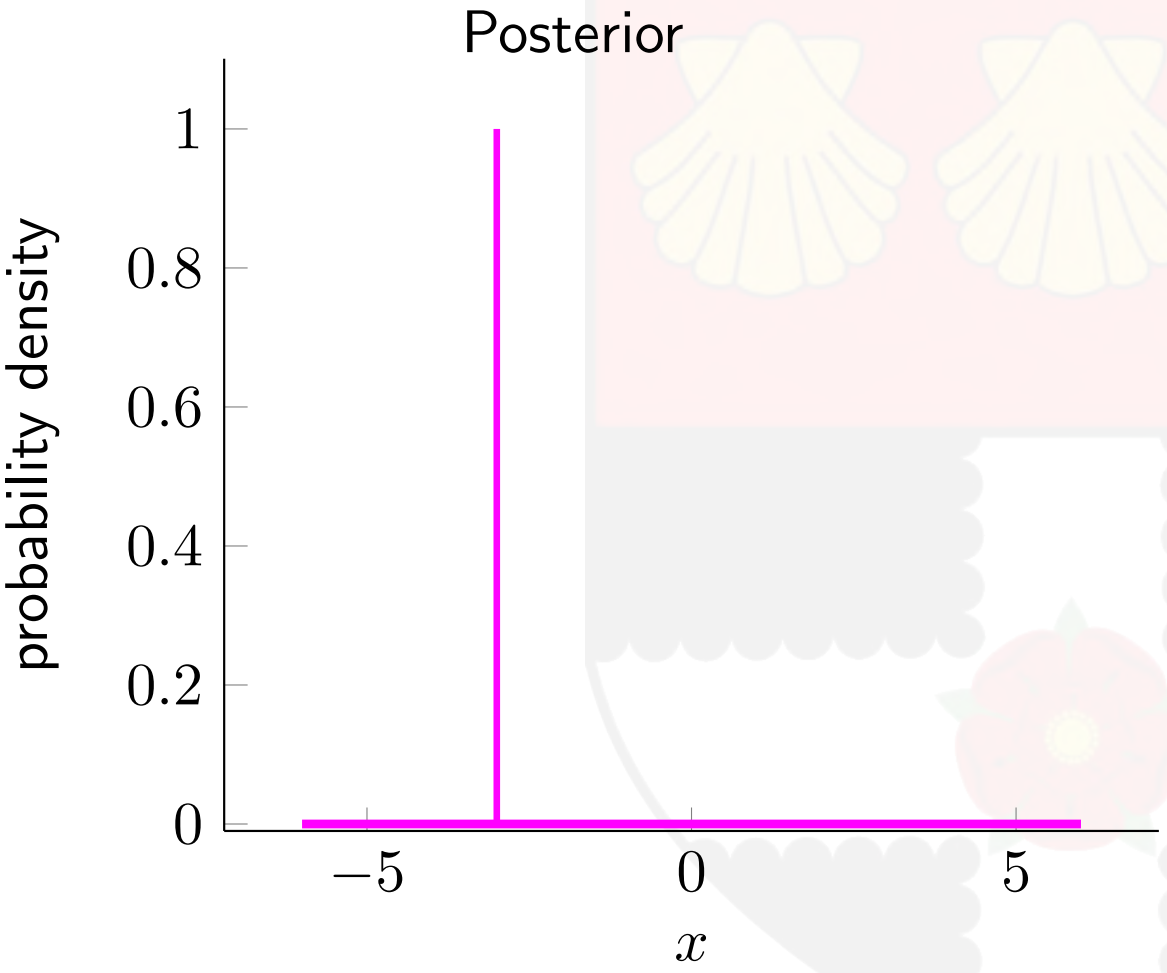

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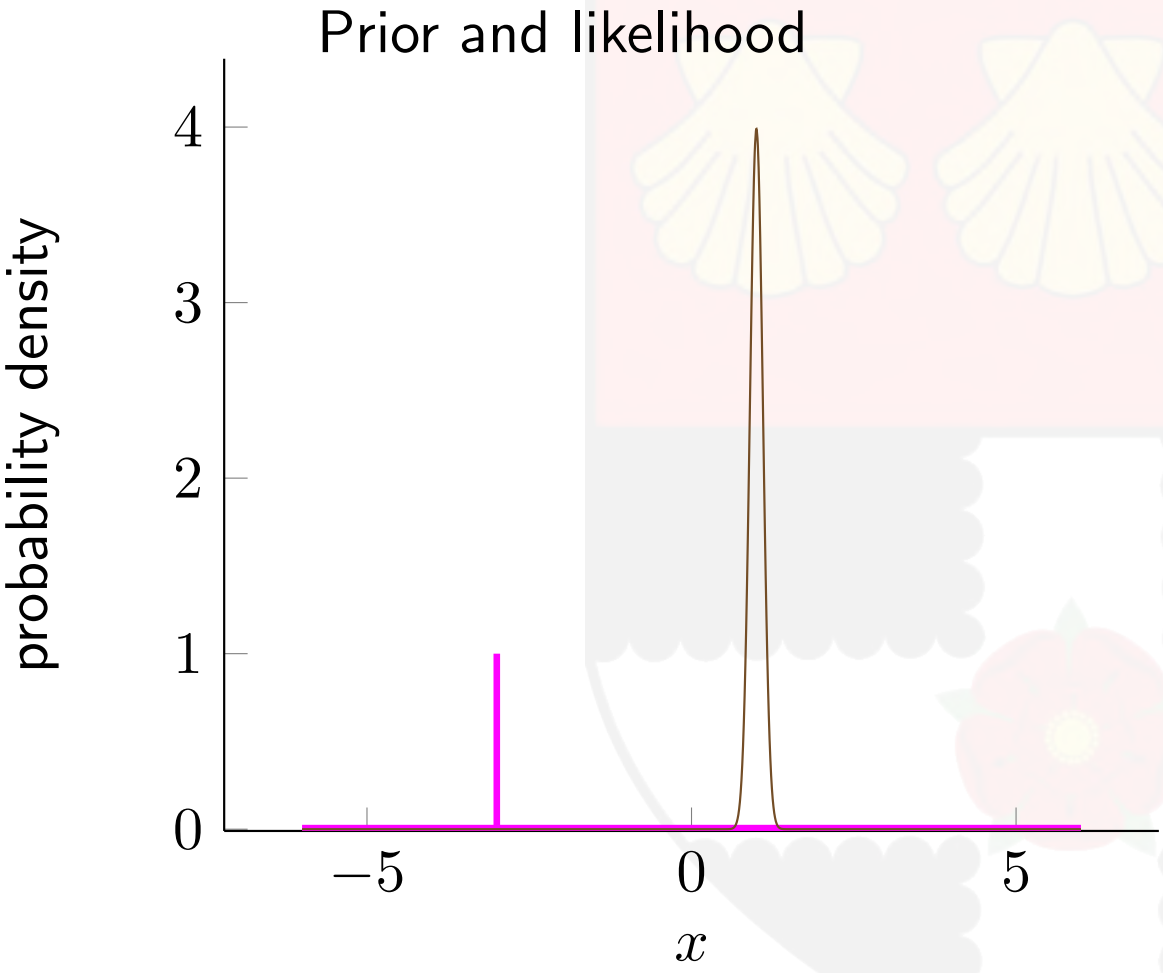

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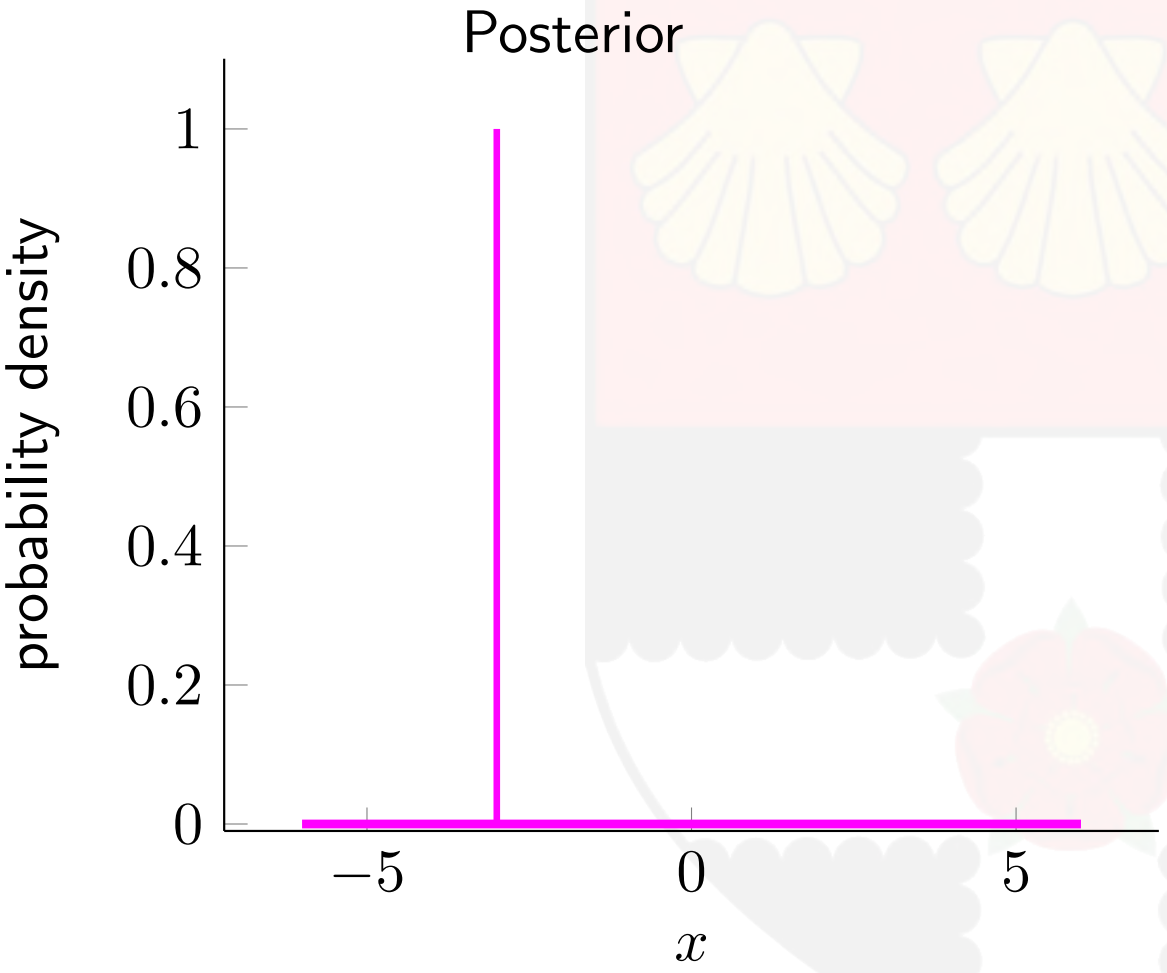

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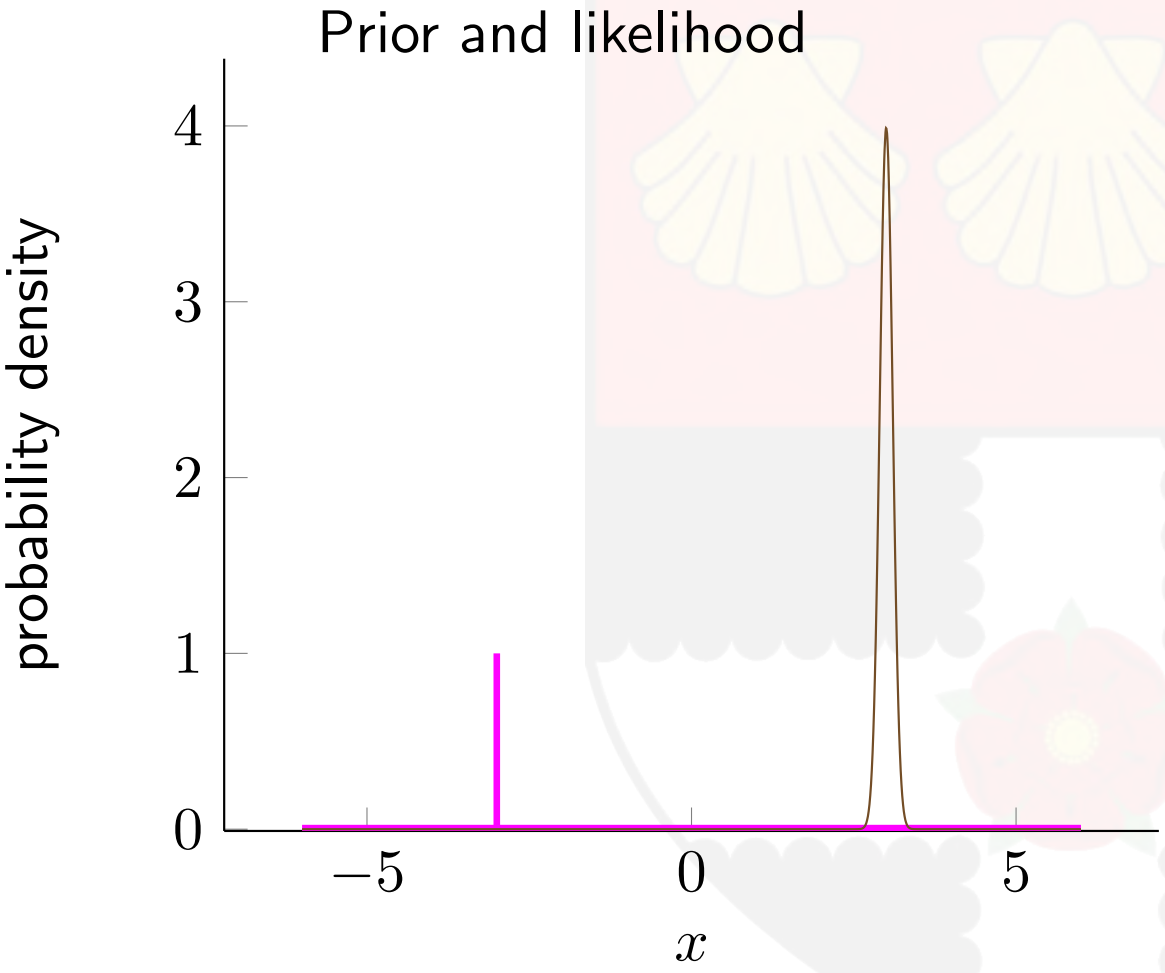

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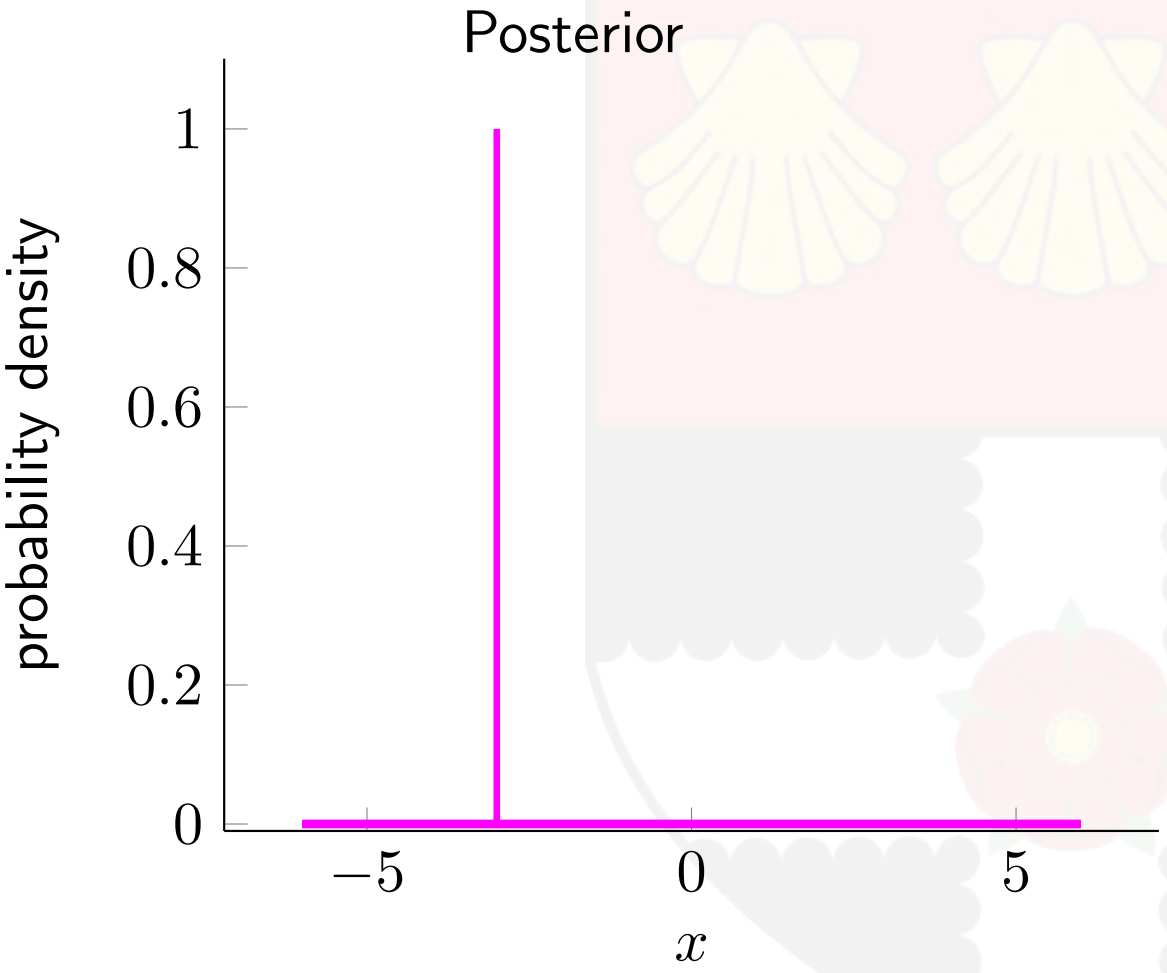


Proposal densities

$$p(x \mid y) = \frac{p(x)p(y \mid x)q(x,y,z)}{p(y)q(x,y,z)}$$

Write the proposal pdf as a sum of delta functions

$$q(x,y,z) = \sum_{i=1}^{M} w_i \delta(x - x_i) \tag{8}$$

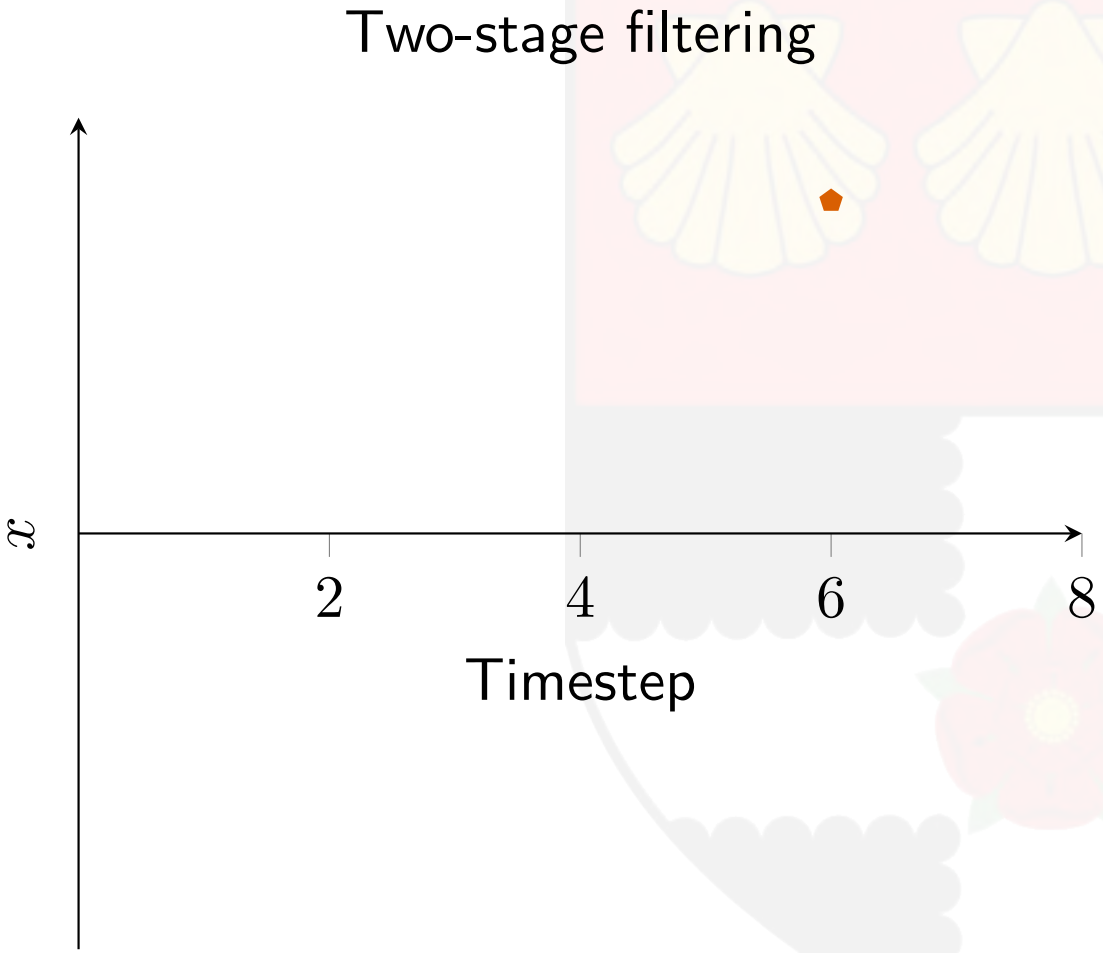
Arrive at a new delta function representation of the posterior

$$p(x \mid y) = \sum_{i=1}^{M} \omega_i \delta(x - x_i) \tag{9}$$

such that

$$\omega_i = \frac{w_i p(x_i) p(y \mid x_i)}{q(x_i, y, z)} \tag{10}$$

Two-stage filters 1



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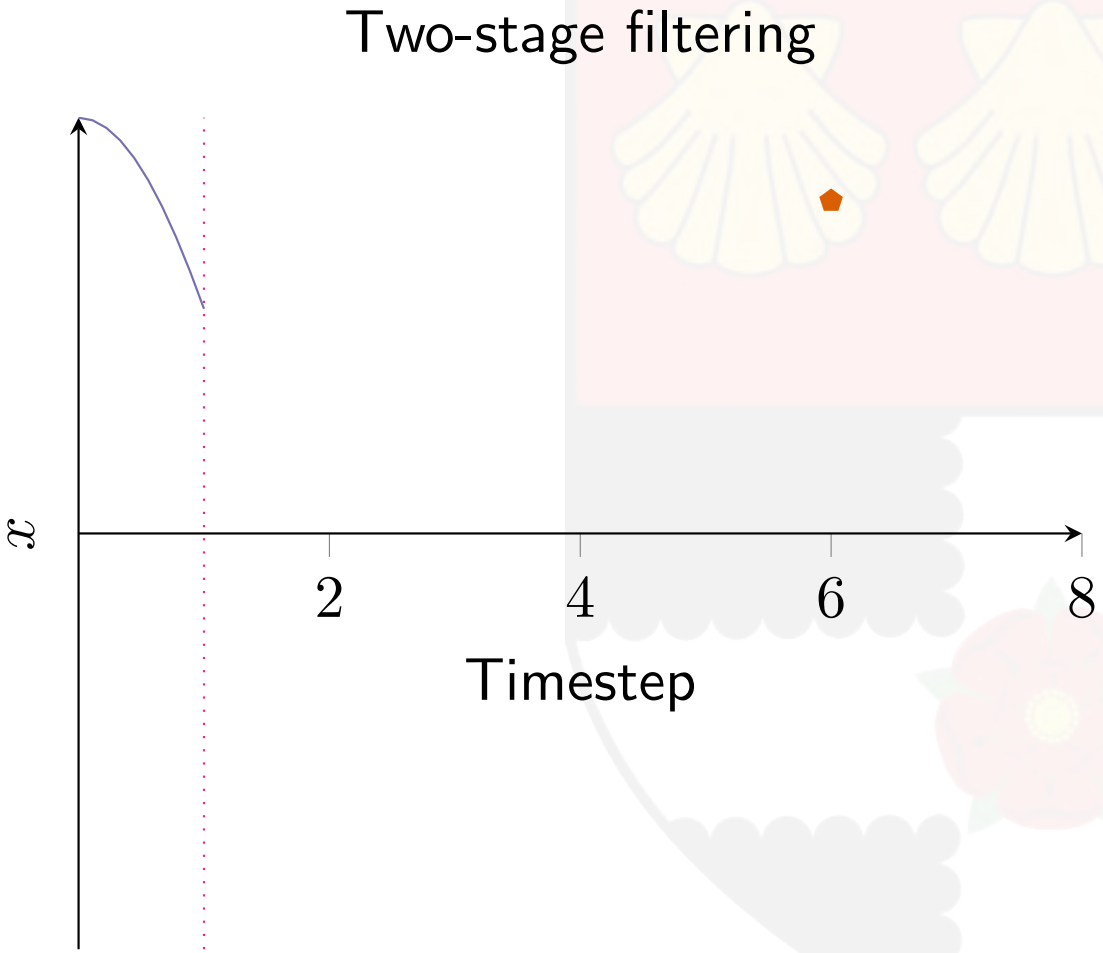

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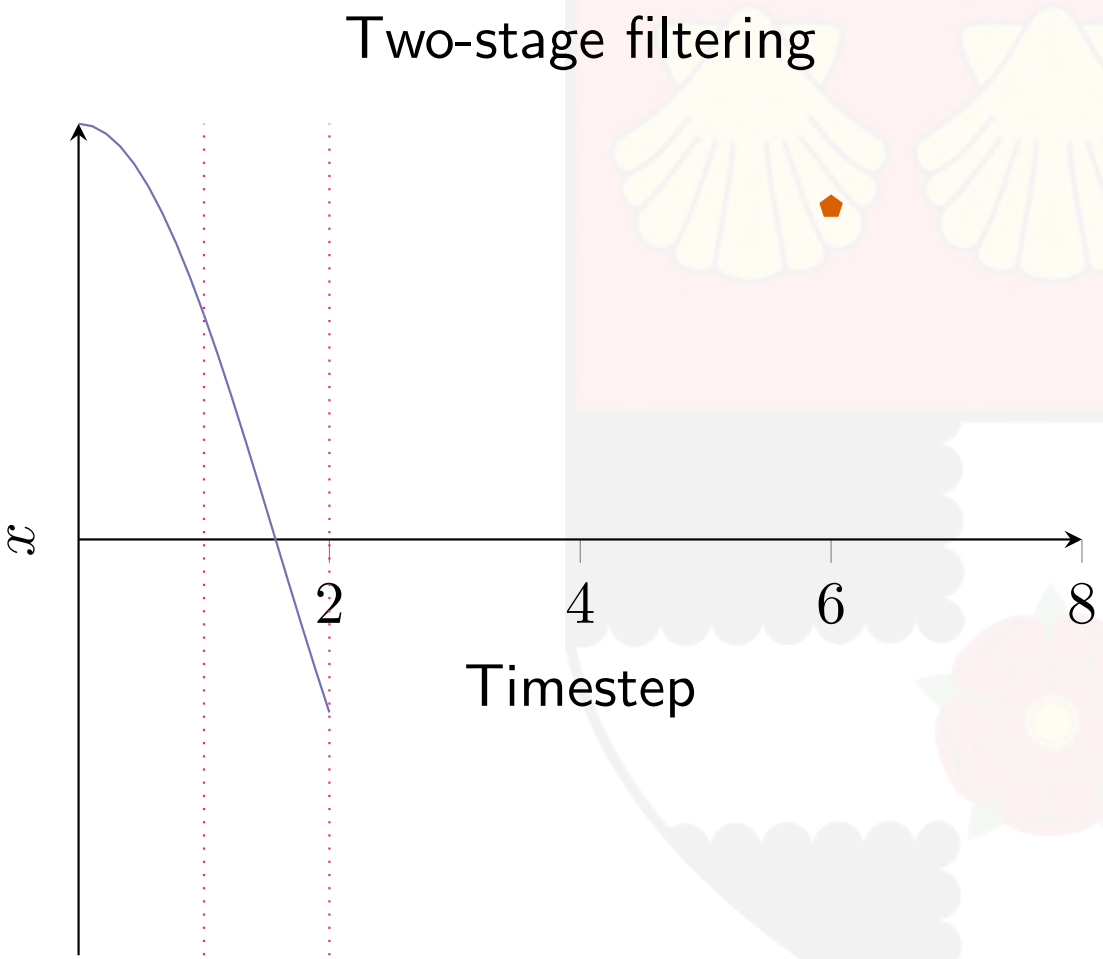

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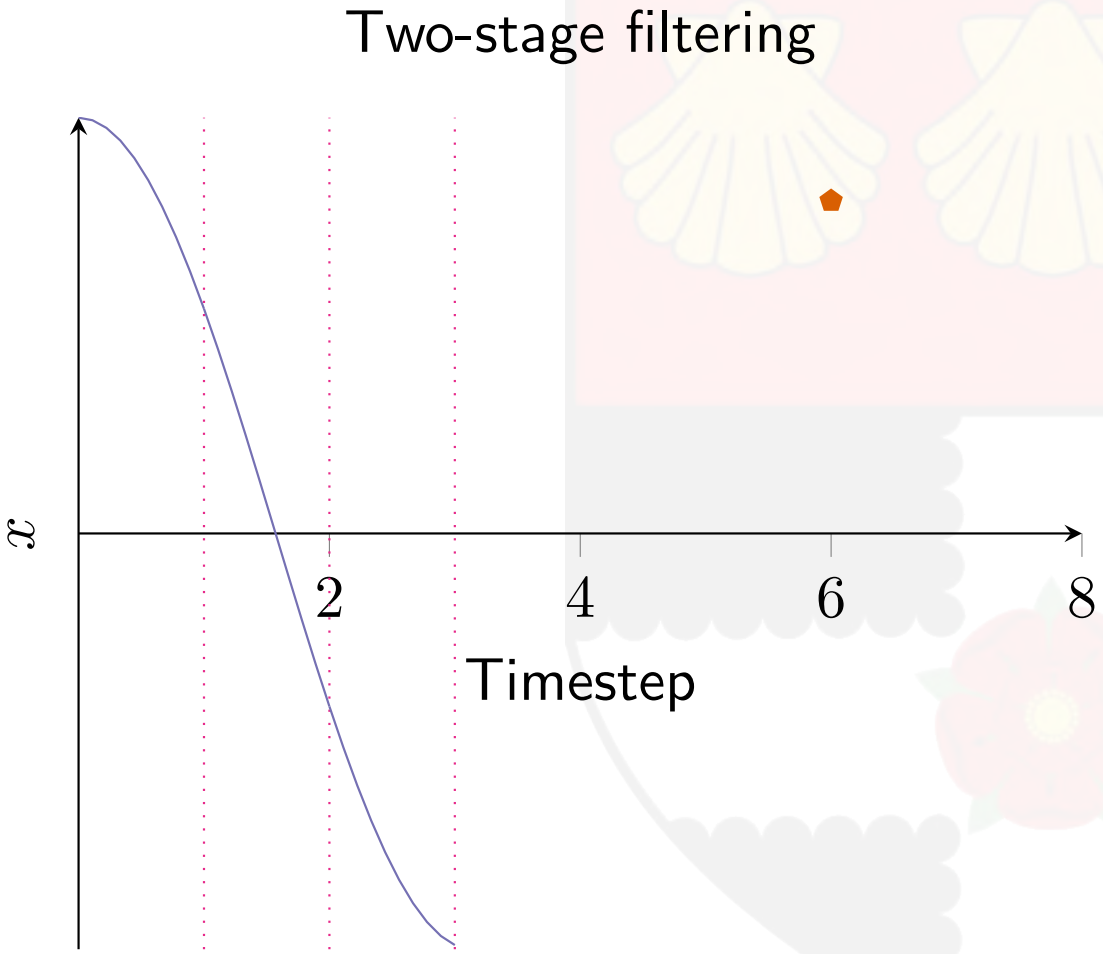

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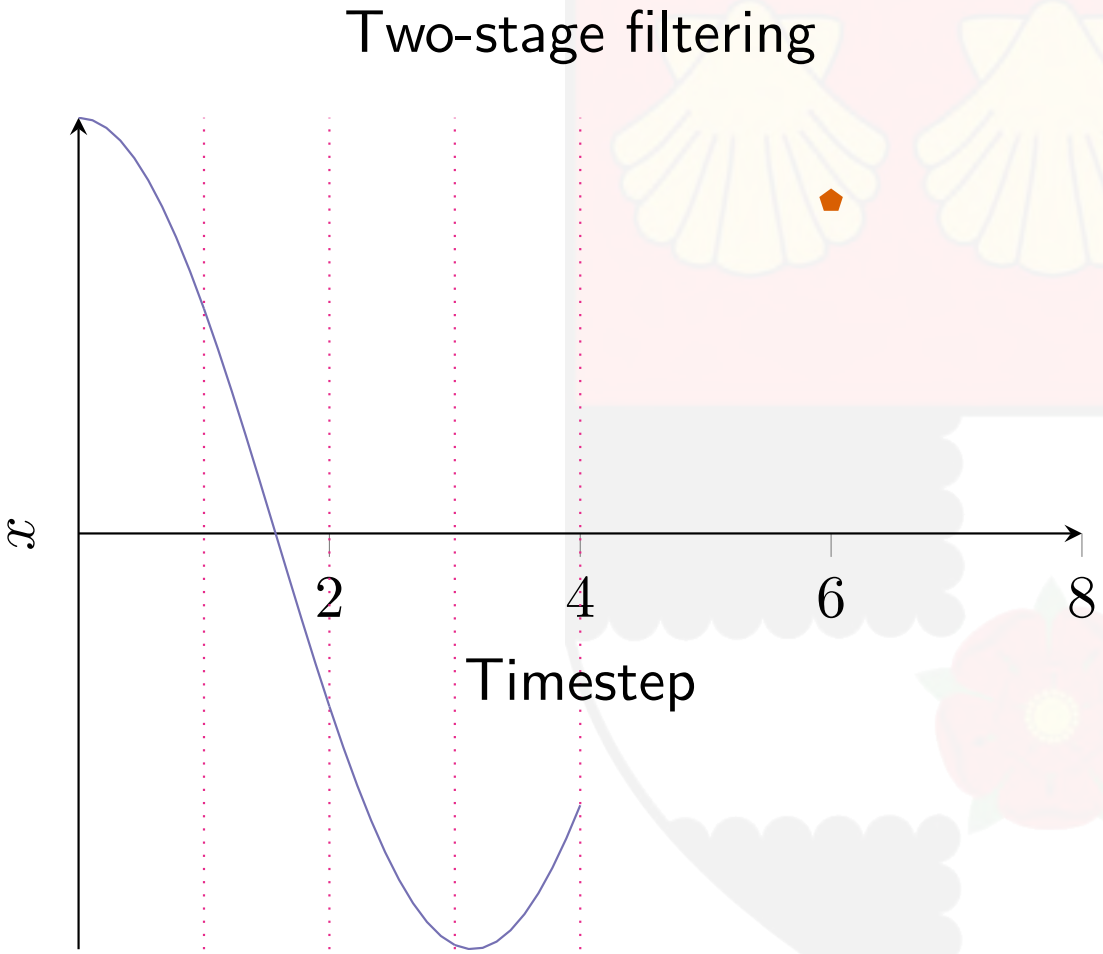

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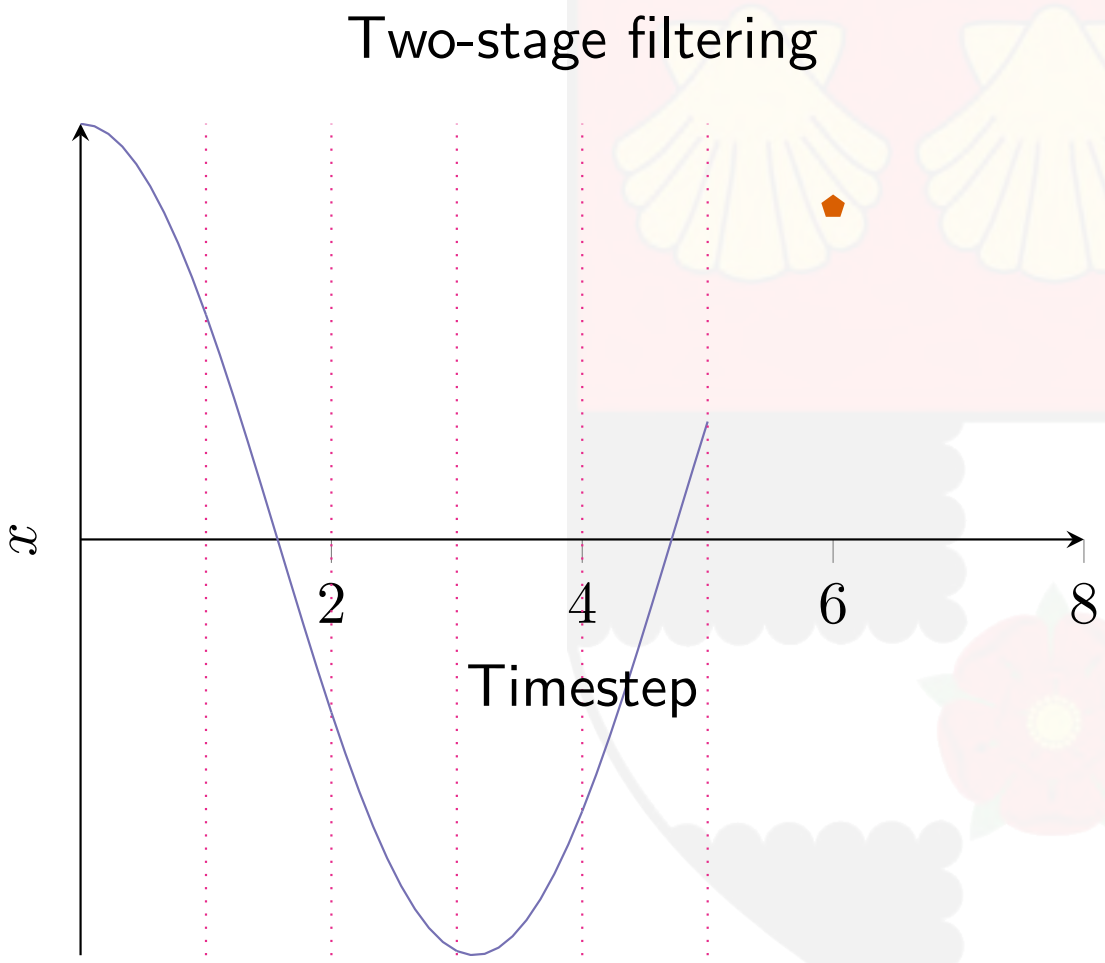

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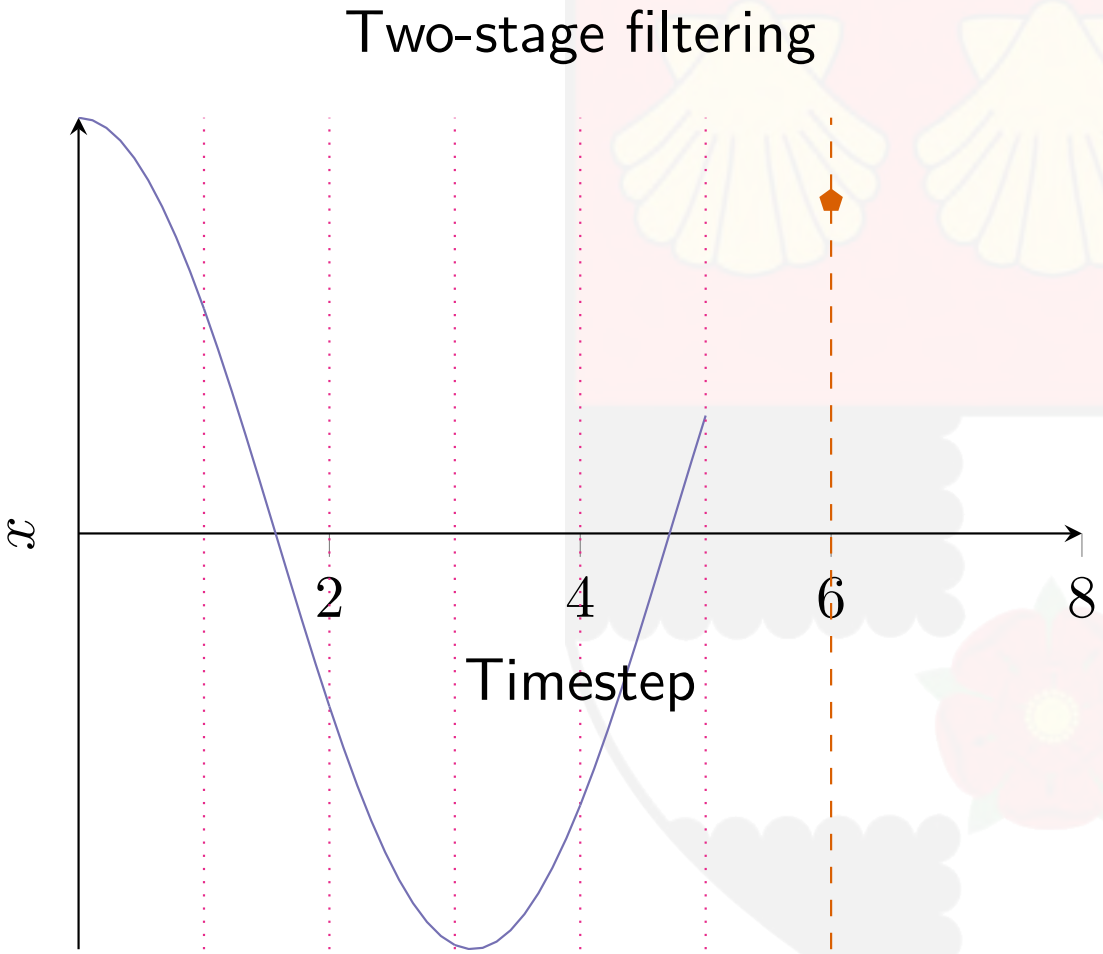

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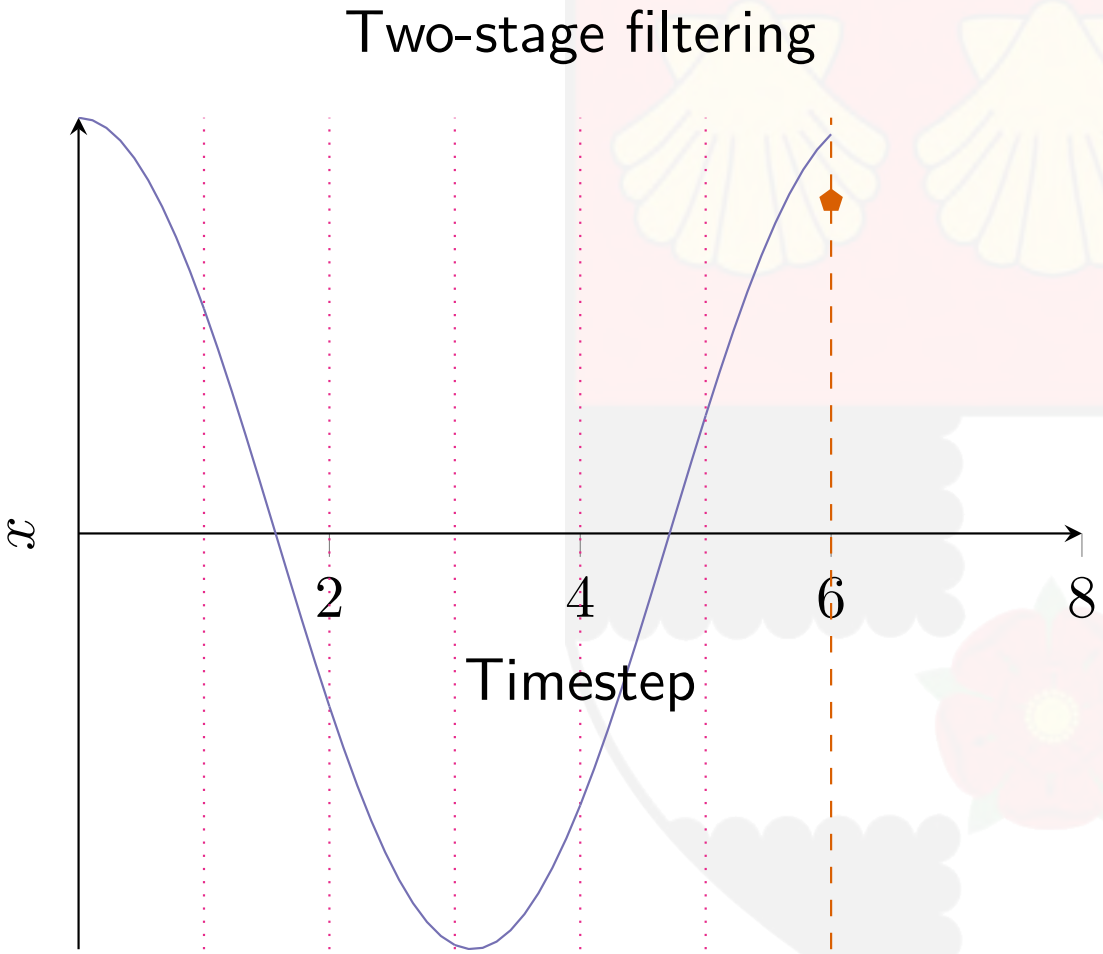

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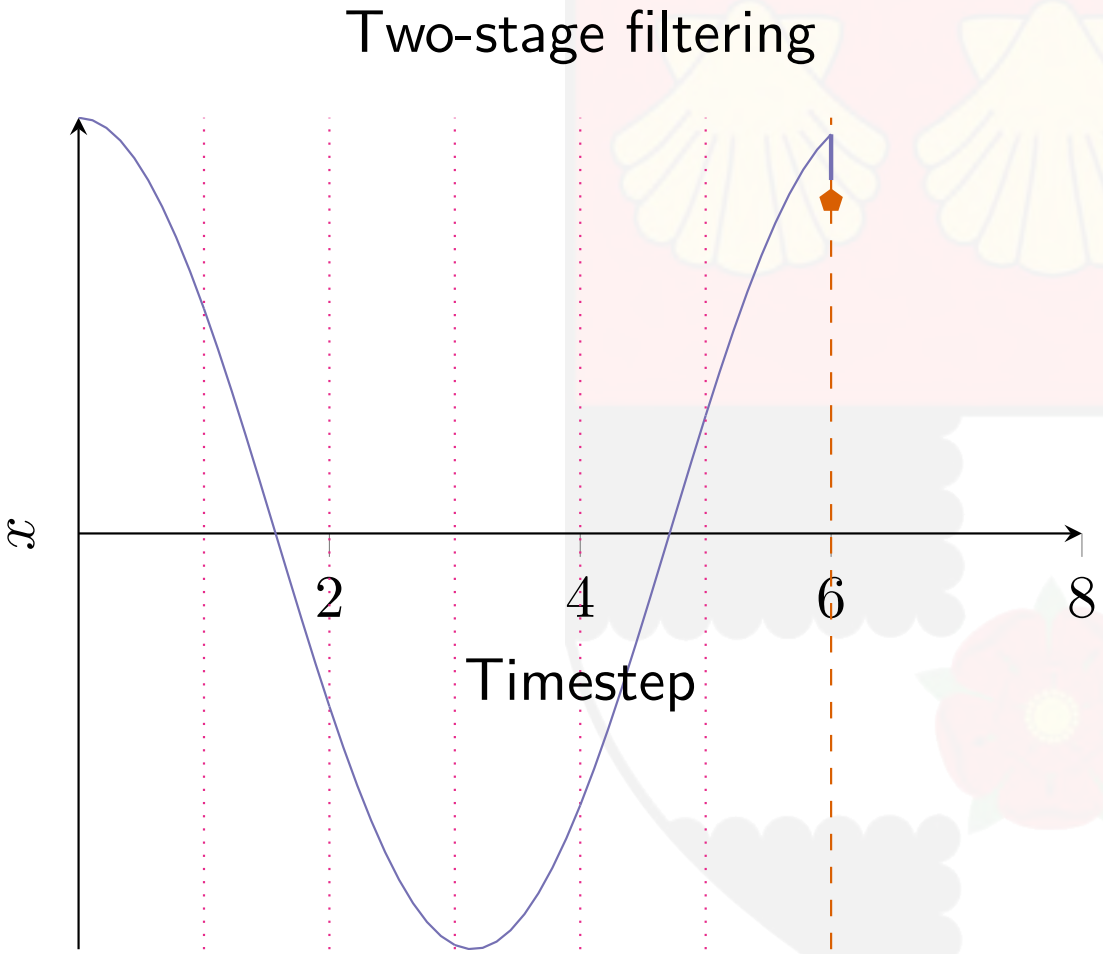

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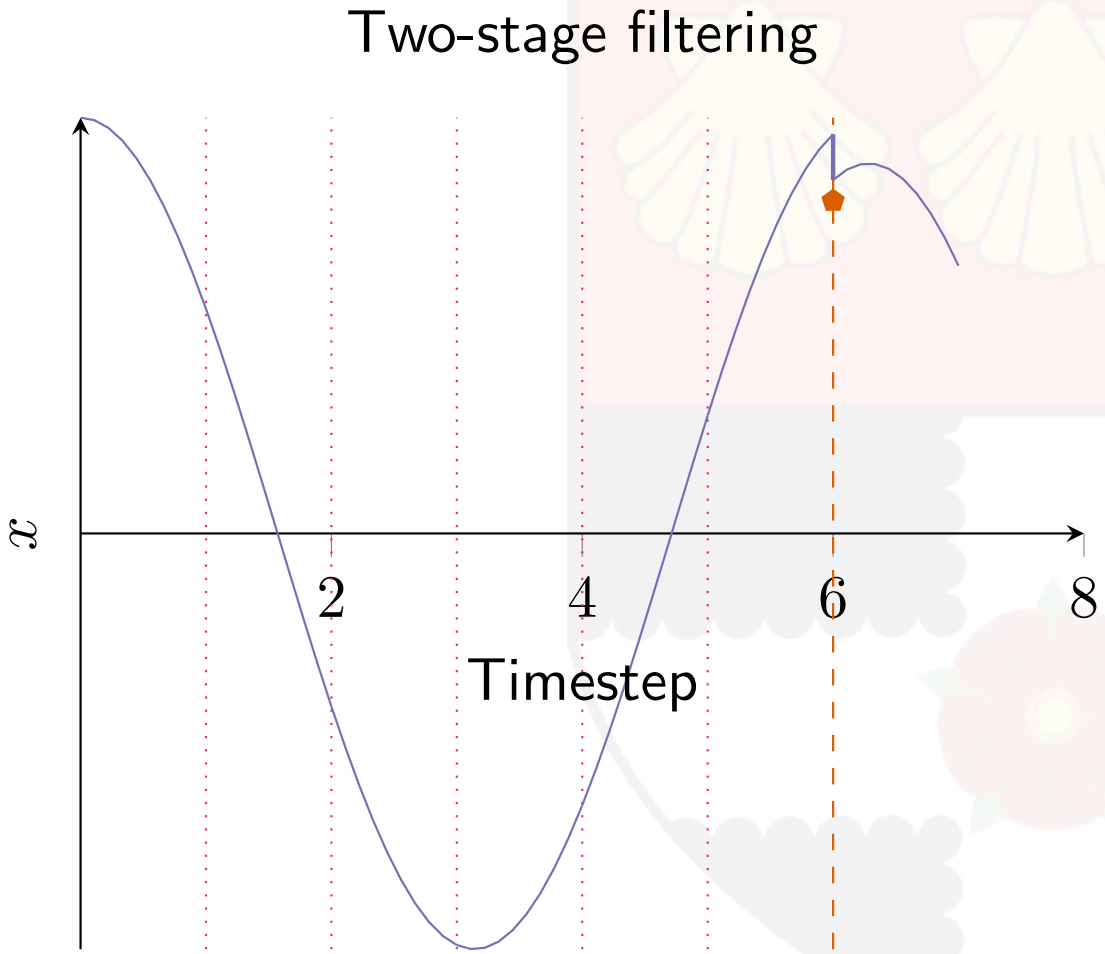

Parameter estimation

Filter degeneracy, a simple example

Suppose we have a model with only a single variable:

$$x = T \tag{11}$$

where T is the temperature in this room, and evolves with Newton's law of cooling:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -K(x - x_r) \tag{12}$$

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Suppose we have N_y thermometers, independent, each that makes Gaussian errors with zero mean and variance σ_r^2.

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Take the measurements, and they all agree that the temperature is $300K$.

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Suppose we have N_y thermometers, independent, each that makes Gaussian errors with zero mean and variance σ_r^2.
Take the measurements, and they all agree that the temperature is $300K$.
Start with an initial prior ensemble, evolve for a few days and look at two different ensemble members.

$$x_1 = 300 + \sigma_r \qquad\qquad x_2 = 300 - 2\sigma_r \tag{13}$$

Filter degeneracy, a simple example

Gaussian likelihood:

$$\omega_i = p(y \mid x_i) = \exp\left[-\frac{1}{2}(y - H(x_i))^T R^{-1}(y - H(x_i))\right] \quad (14)$$

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$$\frac{\omega_1}{\omega_2} = \frac{\exp\left[-\frac{1}{2}(y - H(x_1))^T R^{-1}(y - H(x_1))\right]}{\exp\left[-\frac{1}{2}(y - H(x_2))^T R^{-1}(y - H(x_2))\right]} = \frac{\exp\left[-\frac{1}{2}\sigma_r N_y \sigma_r^{-2} \sigma_r\right]}{\exp\left[-\frac{1}{2} 2\sigma_r N_y \sigma_r^{-2} 2\sigma_r\right]}$$

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$$\frac{\omega_1}{\omega_2} = \frac{\exp\left[-\frac{1}{2}N_y\right]}{\exp\left[-2N_y\right]} = \exp\left[\frac{3}{2}N_y\right] \quad (15)$$

Parameter estimation via state augmentation

Let's use the Lorenz 63 example:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y - x)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = x(r - z) - y$$
$$\frac{\mathrm{d}z}{\mathrm{d}t} = xy - bz$$

State estimation

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

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State estimation

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Joint state and parameter estimation

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ \sigma \\ r \\ b \end{bmatrix}$$

Parameter estimation using the EnKF

What model should we use?

$$\mathbf{x}^{k+1} = f(\mathbf{x}^k)$$

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Joint state and parameter estimation

$$\begin{bmatrix} x^{k+1} \\ y^{k+1} \\ z^{k+1} \\ \sigma^{k+1} \\ r^{k+1} \\ b^{k+1} \end{bmatrix} = \begin{bmatrix} f\left(\begin{bmatrix} x^k \\ y^k \\ z^k \end{bmatrix}\right) \\ \sigma^k \\ r^k \\ b^k \end{bmatrix}$$

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Joint state and parameter estimation

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